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Smaller thermal models that preserve the physics.

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Abstract

This paper derives a structure-preserving reduction for a linear grounded thermal network. A full-column-rank Galerkin basis preserves positive thermal capacitance and the positive-definite conductance matrix, giving a reduced model with decaying unforced dynamics. For a partition-based aggregation, total heat capacity and the aggregate heat balance are retained as well. A specified three-node example is reduced to two states and evaluated analytically: both models remain stable, but the reconstructed source temperature has a one-kelvin steady-state error under a ten-watt input. The distinction between stability, heat balance and output accuracy provides an explicit acceptance framework for compact thermal models.

1 The reduction problem

A thermal model used repeatedly for estimation or optimization may contain more states than the task requires. Model reduction replaces these states with a smaller coordinate vector. Its objective should specify the temperature outputs, operating range and prediction horizon that matter. A small reconstruction error averaged over all states is not automatically a small error at the heat source.

Projection methods provide a systematic framework for this replacement [1]. Here the full model is a linear conductive network with constant parameters and temperature rises θ relative to a fixed bath. C is a diagonal matrix of strictly positive heat capacities in J/K. K is the symmetric conductance matrix in W/K. Each connected component has a path to the bath, making K positive definite.

$$C\dot{\boldsymbol{\theta}} = -K\boldsymbol{\theta} + Bu, \qquad y = H\boldsymbol{\theta}. \tag{1}$$

Bu is the nodal heat input in watts and H selects the temperature outputs.

2 Preserve the matrices before simplifying the states

Approximate θ by Vz, where V has independent columns and z contains fewer coordinates. Requiring the residual to be orthogonal to the columns of V gives $C_r = V^{\mathsf{T}}CV$, $K_r = V^{\mathsf{T}}KV$ and $B_r = V^{\mathsf{T}}B$. For any nonzero reduced vector z, Vz is nonzero; hence $z^{\mathsf{T}}C_rz$ and $z^{\mathsf{T}}K_rz$ are positive. The positive-definite structure survives the projection.

Consequently -$C_r{}^{-1}K_r$ is similar to the symmetric negative-definite matrix -C_r^(-1/2) K_r C_r^(-1/2). All its eigenvalues are negative. This establishes asymptotic decay for the unforced continuous-time reduced model. It does not establish stability of an arbitrary feedback controller connected to that model, and it does not address changes in flow or boundary conditions outside the parameter range.

$$C_r\dot{z} = -K_rz + B_ru, \qquad C_r = V^{\mathsf{T}}CV, \quad K_r = V^{\mathsf{T}}KV. \tag{2}$$

A congruence projection preserves positive definiteness when V has full column rank.

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3 Dissipation is not the same quantity as stored heat

The quadratic functional W = $\frac{1}{2}\theta^{\mathsf{T}}C\theta$ gives dW/dt = -θ^{T}Kθ + θ^{T}Bu. It is a useful mathematical dissipation certificate, consistent with structure-preserving reduction principles [2]. W has units J·K; it is not the physical sensible heat stored relative to the bath. That heat is Q = 1^{T}Cθ, with units of joules. Confusing the two quantities obscures what a reduction actually preserves.

For a partition aggregation, every full node belongs to exactly one reduced group. The columns of V are group indicators and V$1_r = 1_n$. Multiplying the projected balance by $1_r{}^{\mathsf{T}}$ reproduces the full aggregate heat balance evaluated at θ = Vz. The reduced capacitance of a group equals the sum of its member capacitances. A general modal basis need not have this direct temperature-group interpretation.

$$Q = \mathbf{1}_n^{\mathsf{T}} C\boldsymbol{\theta}, \qquad W = \tfrac{1}{2}\boldsymbol{\theta}^{\mathsf{T}} C\boldsymbol{\theta}, \qquad \dot{W} = -\boldsymbol{\theta}^{\mathsf{T}} K\boldsymbol{\theta} + \boldsymbol{\theta}^{\mathsf{T}} Bu. \tag{3}$$

Q is physical sensible heat in J; W is a quadratic mathematical functional in J·K.

4 A three-node network reduced to two states

Consider capacities of 10, 20 and 30 J/K. Conductances are 10 W/K between nodes one and two, 5 W/K between nodes two and three, and 2 W/K from node three to the bath. Apply 10 W at node one, starting from zero temperature rise. Aggregate the first two nodes by V = [[1, 0], [1, 0], [0, 1]]. The reduced capacities are 30 and 30 J/K, and K_r = [[5, -5], [-5, 7]] W/K.

The full decay rates are approximately -1.605764, -0.347720 and -0.029850 s^{-1}. The reduced rates are -0.369967 and -0.030033 s^{-1}. Both systems are stable. At steady state, however, the full temperature rises are [8, 7, 5] K, while the reconstructed reduced state is [7, 7, 5] K. Aggregation removes the one-kelvin gradient required to carry 10 W through the first conductance. A stable model can therefore miss the heat-source temperature.

Preserving the aggregate balance is not a claim that the approximate trajectory stores exactly the same heat as the full trajectory. At steady state the full example stores 10×8 + 20×7 + 30×5 = 370 J relative to the bath. Its reduced reconstruction stores 30×7 + 30×5 = 360 J. For a nonuniform initial temperature, an energy-consistent group initialization is the capacitance-weighted average: z_0 = $C_r{}^{-1}V^{\mathsf{T}}C\theta_0$. This preserves initial total sensible heat for this partition, but cannot retain every internal gradient.

$$C = \mathrm{diag}(10, 20, 30)\ \mathrm{J/K}, \qquad K = \begin{bmatrix} 10 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 7 \end{bmatrix}\ \mathrm{W/K}. \tag{4}$$

All parameters are illustrative. Responses are calculated from the constant linear systems.

Table 1: Calculated source-temperature rise after the illustrative 10 W step.

Time / s	**Full model / K**	**Reduced model / K**	**Reduced minus full / K**
1	0.676	0.308	-0.367
5	1.848	1.231	-0.617
10	2.777	2.075	-0.702
30	5.136	4.304	-0.832
∞	8.000	7.000	-1.000

5 Accept a reduction against its outputs

The example is a counterexample to using stability or conserved capacity as a complete accuracy criterion. The relevant error is y - y_r = Hθ - HVz. Compare this output on held-out power profiles, initial temperatures and boundary conditions. Include peak error and the timing of excursions as well as an average metric. State the units, test duration, sampling interval and the tested parameter range.

A suitable basis can retain a source-local state while aggregating more slowly varying regions. Alternatively, select modes or interpolation conditions that emphasize the required input-output map. Established projection frameworks offer several such choices [1]. Their performance must be assessed for the chosen outputs; there is no general rule that three states capture a fixed percentage of thermal information.

6 Numerical and operating limits

Continuous-time stability also does not guarantee that every numerical time step is stable. For a real negative mode λ, explicit Euler requires $|1 + h\lambda| < 1$. The critical step is $2/|\lambda$fast$|$: approximately 1.246 s for the full model and 5.406 s for the reduced model. The step must be strictly below the respective exact critical value. These are stability limits of the specified linear models, not accuracy recommendations. Backward Euler preserves decay for this network for any positive step, but large steps can still miss important transients.

Variable-flow convection, advection and nonlinear heat transfer require a revised model and a corresponding structure analysis. A fixed basis may remain dissipative across positive conductance changes while losing output accuracy. The final acceptance decision must therefore keep three questions separate: does the network satisfy the intended heat balance, does its numerical realization decay appropriately, and does it predict the required temperature with sufficient accuracy?

Conclusion

Preserve dissipation by construction. Establish temperature accuracy with an explicit input-output comparison.

References

[1] Benner, Gugercin & Willcox (2015). A Survey of Projection-Based Model Reduction Methods for Parametric Dynamical Systems. SIAM Review 57(4), 483-531. DOI:10.1137/130932715.. `https://doi.org/10.1137/130932715`.

[2] Gugercin, Polyuga, Beattie & van der Schaft (2012). Structure-preserving tangential interpolation for model reduction of port-Hamiltonian systems. Automatica 48(9), 1963-1974. DOI:10.1016/j.automatica.2012.05.052.. `https://arxiv.org/abs/1101.3485`.